Binary Response Models

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2021

Chapter Summary

The first departure from the CLRM we will discuss are the Binary Response Models. We use this when the dependent variable is some dummy. We will show that the mere usage of the OLS (i.e. LPM) is heavily biased. However, interpreting the coefficients in the appropriate models such as the Logit and Probit is not always straightforward.

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$$Feeder_i = \beta_0 + \beta_1 \ln(ParentWage_i) + u_i$$

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Therefore

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- β_1 is merely the change in the probability that $y = 1$ when x changes.

$$\Delta P(y = 1|x)|_{x \to x+1} = \beta_1$$

Circling back to Feeder

Relating it to our example on feeder schools

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Answer: No! Let's discuss three reasons why not

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- This is nonsensical! We cannot have predicted values like that.

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$$u_i = \begin{cases} -\beta x_i & \text{if } y_i = 0 \\ 1 - \beta x_i & \text{if } y_i = 1 \end{cases}$$

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- Let $P_i = \beta x_i = P(y_i = 1|x_i)$. Also note that $P(y_i = 0|x_i) + P(y_i = 1|x_i) = 1$. Therefore $P(y_i = 0|x_i) = 1 - P(y_i = 1|x_i)$

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$$Var(u_i|x_i) = (1 - P_i) \cdot (-\beta x_i)^2 + P_i \cdot (1 - \beta x_i)^2$$

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- The variance is clearly dependent on i and is not a constant. Therefore, the variance is heteroscedastic (not constant).

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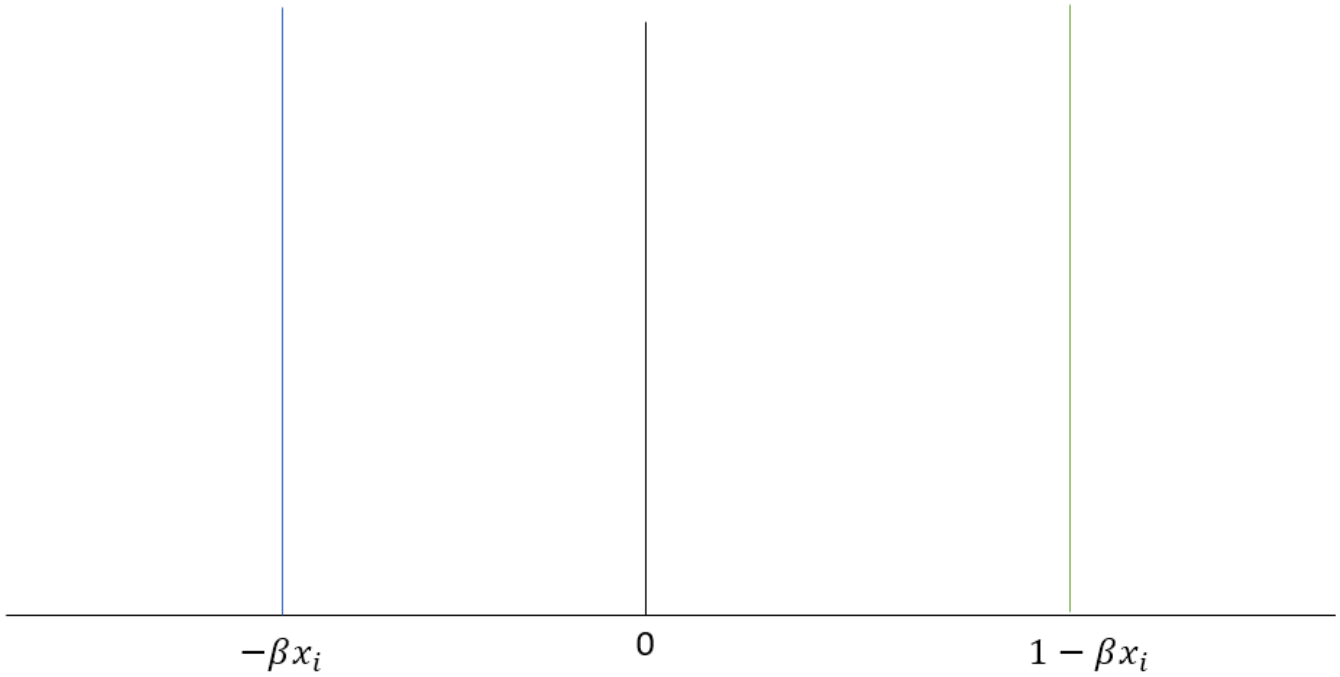

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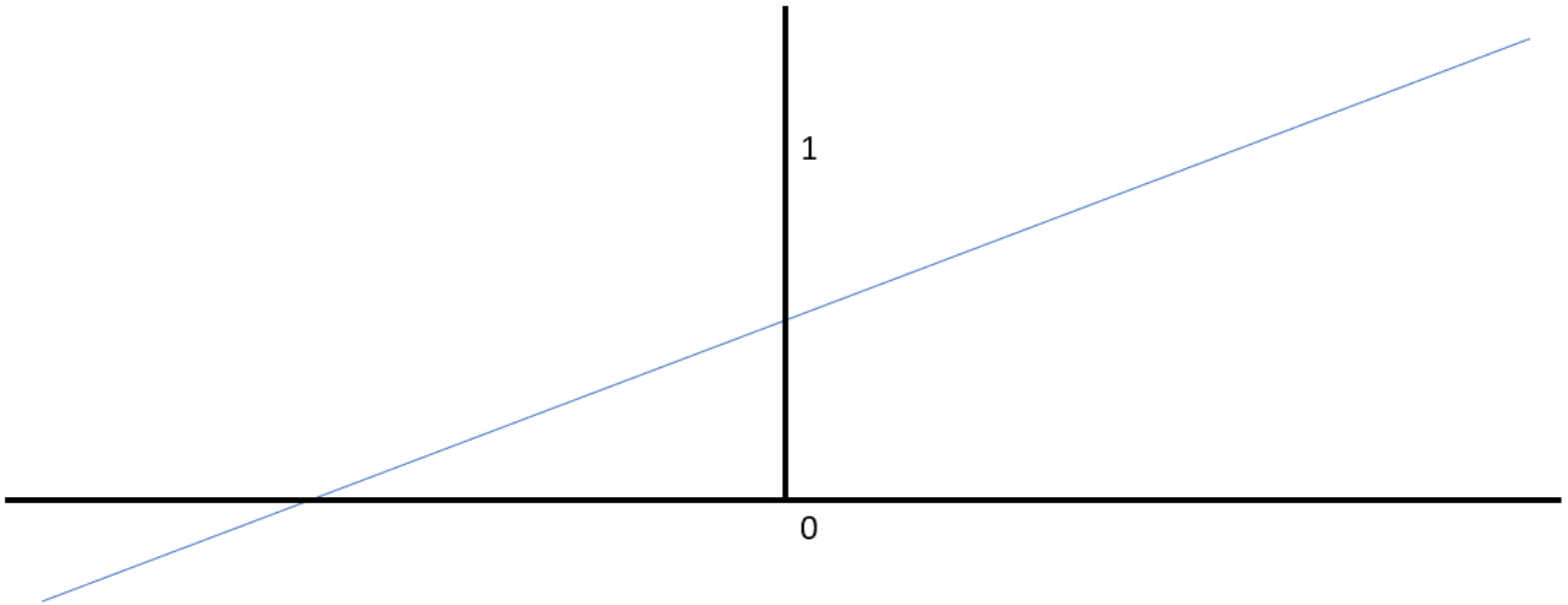

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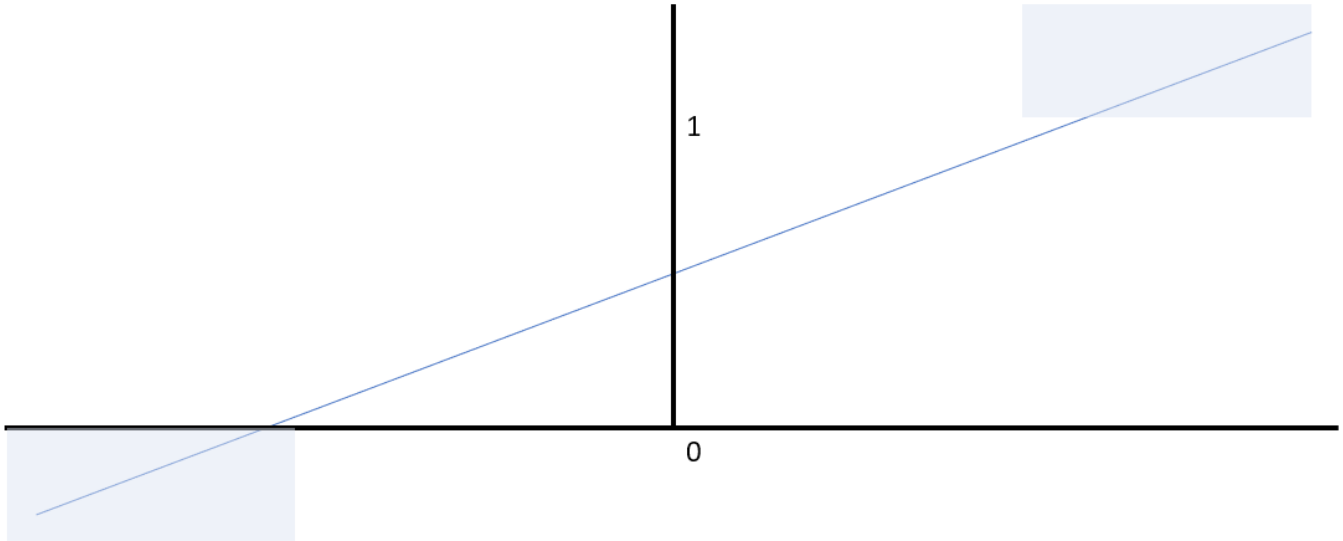

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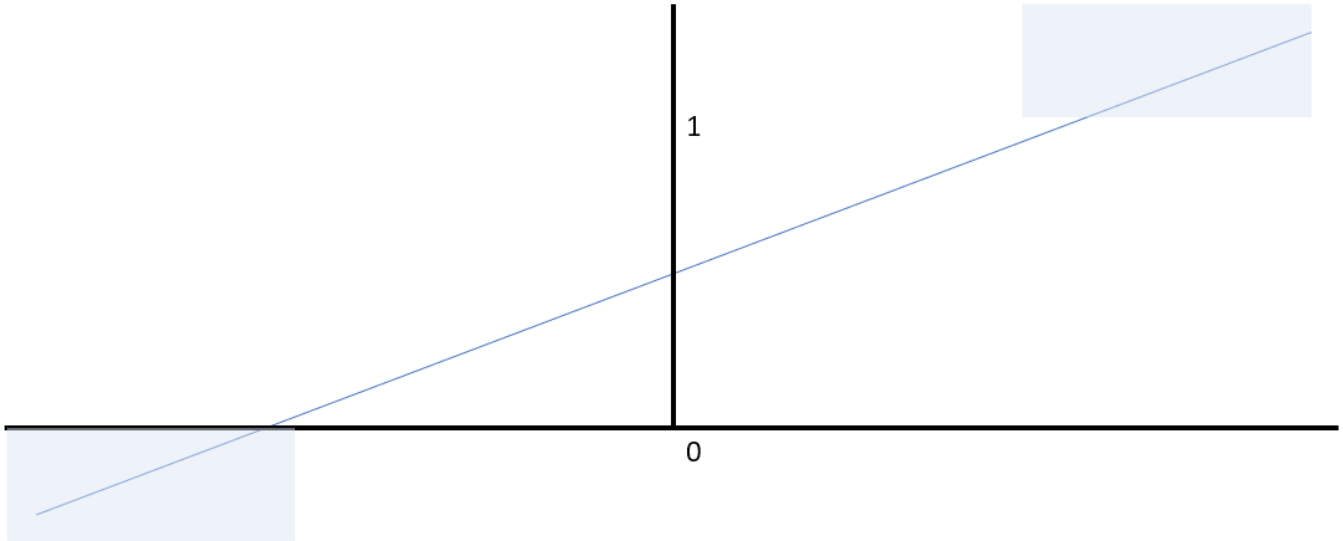

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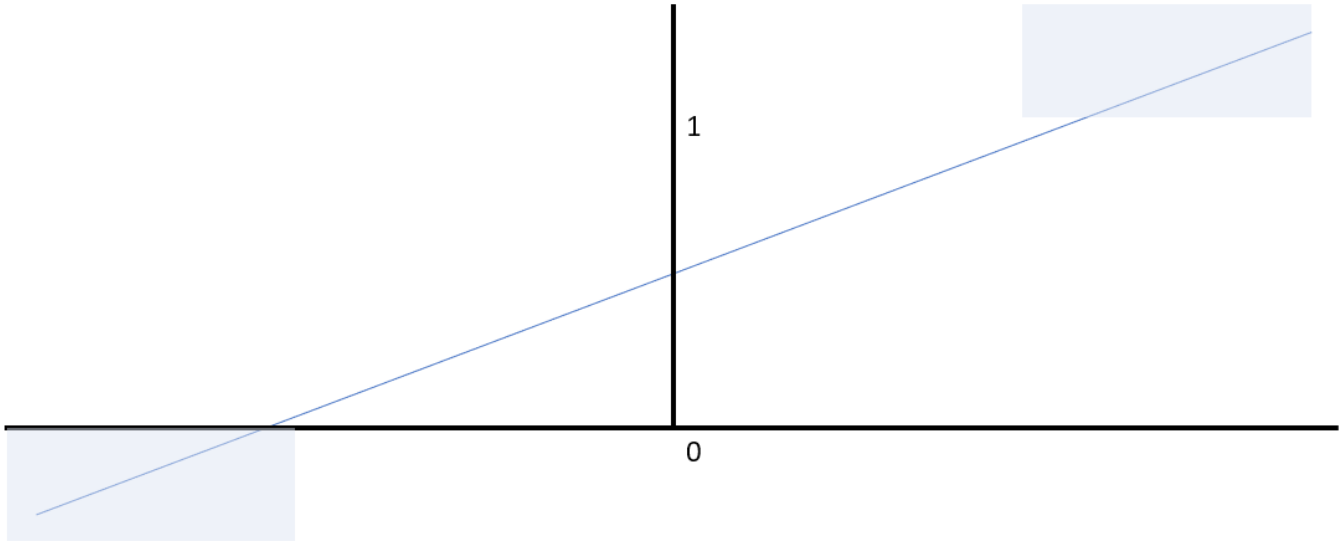

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Idea

We need some way to shrink and force the line to fit between zero and one. That shrinkage will solve the issue and that is precisely what Logit and Probit are.

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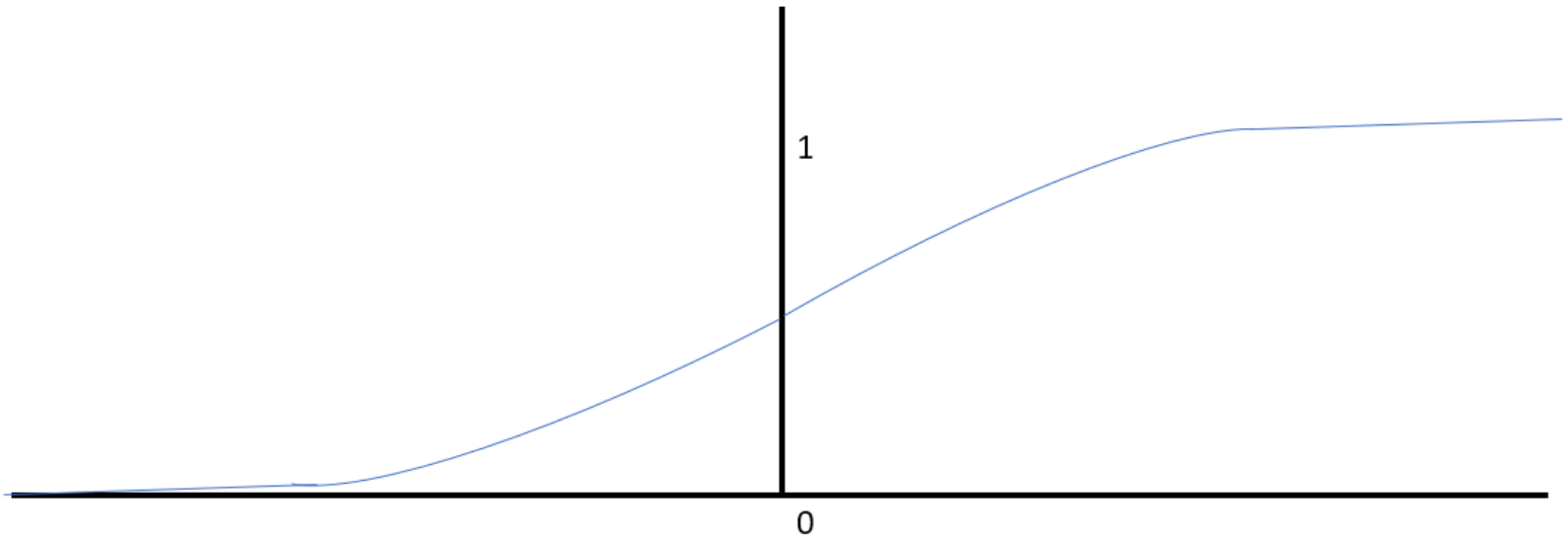

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- Second, as the $z \to +\infty$, the $F(z) \to 1$.

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- Like Logit, we can predict some probability for a predicted y (until z). But since this is a cdf, we know that if we evaluate $F(-\infty)$, this will tend to zero. Conversely, evaluating $F(+\infty) = 1$.